

# Ballistic aggregation in symmetric and non-symmetric flows

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## Abstract

Explicit solutions for ballistic aggregation of dust-like matter, whose particles stick inelastically upon collisions, are considered. This system provides a model of large-scale structure formation in cosmology within the Zel'dovich approximation. In particular we show the equivalence of two different representations of solutions proposed in [14, 5] for a flat 1D flow, extend these representations to cylindrically or spherically symmetric flows, and provide explicit counterexamples showing how exactly these representations break down in the case of non-symmetric flow.

## 1 Introduction

We consider here explicit solutions for ballistic aggregation of dust-like matter, with particles that stick upon collisions absolutely inelastically. This system may be considered as a model for the large structure formation in the Universe within the Zel'dovich approximation [16]. First we briefly recall how this model comes about.

Consider a flat Einstein–de Sitter universe containing massive dust-like matter that moves in a self-consistent gravitational field. This matter participates in two types of motion, the homogeneous cosmological Hubble expansion and the evolution of local perturbations of density and velocity, which leads

to formation of large-scale inhomogeneities.

To describe the latter kind of dynamics in coordinates comoving with the Hubble expansion, Ya. B. Zel'dovich proposed [16] to use a nonlinear function of time, the perturbation growth factor, instead of the cosmological time. In the new time, the gravitational force turns out to be approximately balanced by the ‘Hubble drag’ caused by the continuous expansion of the spatial scale. Within this approximation fluid elements move ballistically, and the evolution of local perturbations is described by the following system of equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (1)$$

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = 0, \quad (2)$$

expressing conservation of respectively mass and momentum. Here  $t$  is the new time parameter,  $\rho(\mathbf{x}, t)$  is the mass density field,  $\mathbf{u}(\mathbf{x}, t)$  is the velocity field,  $\mathbf{x}$  is the spatial position comoving with the Hubble expansion, and  $\mathbf{u} \otimes \mathbf{u}$  denotes the tensor with components  $u_i u_j$ .

When trajectories of fluid elements cross, the velocity field ceases to be single-valued, and several streams passing through the same spatial location form. Numerical simulations show (see e.g. [11, 15]) that the gravitational interaction between streams confines the matter to domains of relatively small width and high density (Fig. 1 on page 2), causing it to aggregate in wall-like, filament-like, and cluster-like structures. This process is called *ballistic aggregation*.

In [7, 8] (see also the monograph [9] and [6]) the Burgers equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \mu \Delta \mathbf{u}$$

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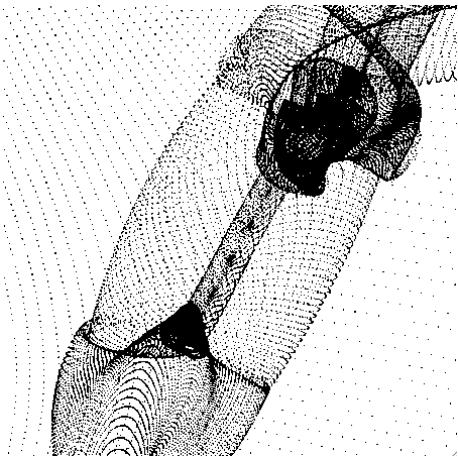


Figure 1: Matter concentration in regions of multi-streamed flow (2D numerical simulations of [11]).

was proposed to describe ballistic aggregation. Indeed, in the vanishing viscosity limit  $\mu \rightarrow 0$  solutions to the Burgers equation develop singularities in the form of infinitesimally thin walls, filaments, and point clusters, whose shape is tantalizingly similar to the large-scale cosmological structure. However this agreement is limited, as the Burgers equation is written in terms of the velocity field and fails to account for conservation of mass or momentum. Quantitatively, direct  $N$ -body simulations of self-gravitating matter disagree with solutions of the Burgers equation for identical initial data.

These limitations can be overcome in the case of 1D flow with flat symmetry (note that in the 1D case a thorough analytical study is possible even without the aggregation approximation [1]). Several authors [14, 10, 5, 3] independently introduced exact solutions for ballistic aggregation in a 1D flow of particles that stick upon collisions with exact conservation of mass and momentum. In particular, in [14] and [5] these solutions were expressed in the form of variational principles, so that the density and velocity fields at any given time could be constructed by minimization of certain functionals.

It is the purpose of the present work to explore the possibility of extending these results to the case where there is no flat symmetry. In Section 2 we

prove the equivalence of the variational principles proposed in [ERS] and [S]<sup>1</sup> in the 1D case with flat symmetry. To make the exposition self-contained we recall the necessary details of constructions of [S] and [ERS], in particular because Ref. [S] is rather difficult to obtain. In Section 3 the variational principles are extended to the case of cylindrical and spherical symmetry.

Both variational principles [S] and [ERS] can be extended formally to the case of 2D or 3D non-symmetric flow, but the corresponding constructions no longer coincide. Moreover, there exist counterexamples showing that neither variational principle is valid in its extended version already for non-symmetric 2D flows. The multidimensional extensions of the variational principles [S] and [ERS] and explicit counterexamples are presented in Section 4.

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## 2 Ballistic aggregation in 1D

### 2.1 Clusters and free particles

In the case of flat symmetry the density field  $\rho(x, t)$  and the velocity field  $u(x, t)$  both depend on a single coordinate. Mass concentration due to inelastic collisions results in the formation of parallel massive ‘walls’ in three-dimensional space, which can be considered as point-like structures (clusters) in the  $x$  axis. We consider first the evolution of a purely discrete mass distribution.

Let  $N$  particles in the  $x$  axis be located initially (i.e., at  $t = 0$ ) at points  $y_1 < y_2 < \dots < y_N$  and have velocities  $u_1, u_2, \dots, u_N$  and masses  $m_1, m_2, \dots, m_N$ . As time passes,  $i$ -th particle describes the trajectory  $x_i(t) = y_i + u_i t$  unless it collides with another particle. After an inelastic collision there forms a cluster containing the whole mass of the group of particles

<sup>1</sup>In the sequel, [14] and [5] are referred to as [S] and [ERS] correspondingly.

that stuck together. Since trajectories of free particles cannot cross, such clusters will always contain groups of particles numbered contiguously, i.e., with  $i^- \leq i \leq i^+$ . Mass, velocity and coordinate of each cluster coincide with the total mass and center-of-mass velocity and coordinate of the corresponding group of particles, defined for the free motion:

$$\begin{aligned} m &= \sum_{i^- \leq i \leq i^+} m_i, \quad u = \frac{1}{m} \sum_{i^- \leq i \leq i^+} u_i m_i, \\ x &= \frac{1}{m} \sum_{i^- \leq i \leq i^+} (y_i + u_i t) m_i. \end{aligned} \quad (3)$$

In order for  $i_0$ -th particle to stay free at time  $t > 0$ , its trajectory must not cross center-of-mass trajectories of any groups of particles immediately adjacent to it. Therefore for any  $i', i''$  such that  $1 \leq i' < i_0 < i'' \leq N$ , the coordinate  $x_{i_0}(t)$  must satisfy the inequalities

$$\begin{aligned} \frac{1}{m'} \sum_{i' \leq i < i_0} (y_i + u_i t) m_i &< x_{i_0}(t) = y_{i_0} + u_{i_0} t \\ &< \frac{1}{m''} \sum_{i_0 < i \leq i''} (y_i + u_i t) m_i, \end{aligned} \quad (4)$$

where

$$m' = \sum_{i' \leq i < i_0} m_i, \quad m'' = \sum_{i_0 < i \leq i''} m_i.$$

Using these conditions and equations (3), one can find coordinates and masses of all particles and clusters at  $t > 0$  without explicitly computing dynamics of the system at intermediate times. Thus equations (4) and (3) give a complete solution for a discrete initial mass distribution.

In the case of a continuous mass distribution the initial density and velocity fields are given by the functions  $\rho_0(x)$  and  $u_0(x)$ . In this case one replaces summation in equations (3) and (4) with integration, the index  $i$  with the continuous initial coordinate  $y$ , and mass of  $i$ -th particle with the mass element  $\rho_0(y) dy$ .

Equation (4) then assumes the form

$$\begin{aligned} \frac{\int_{y'}^{y_0} (y + u_0(y)t) \rho_0(y) dy}{\int_{y'}^{y_0} \rho_0(y) dy} &\leq y_0 + u_0(y_0)t \\ &\leq \frac{\int_{y_0}^{y''} (y + u_0(y)t) \rho_0(y) dy}{\int_{y_0}^{y''} \rho_0(y) dy}. \end{aligned} \quad (5)$$

If these inequalities are satisfied for any  $y' < y_0 < y''$  and at least one of them is strict, the particle located initially at  $y_0$  will stay free at  $t > 0$  and will be located at  $x(y_0, t) = y_0 + u_0(y_0)t$ .

On the other hand, let a group of particles located initially at  $y^- < y < y^+$  stick together at time  $t$  into a cluster surrounded by free particles. Then mass, velocity, and coordinate of this cluster are given by

$$\begin{aligned} m &= \int_{y^-}^{y^+} \rho_0(y) dy, \quad u = \frac{1}{m} \int_{y^-}^{y^+} u_0(y) \rho_0(y) dy, \\ x &= \frac{1}{m} \int_{y^-}^{y^+} (y + u_0(y)t) \rho_0(y) dy. \end{aligned} \quad (6)$$

For any particle specified by the initial position  $y$ , equations (5) and (6) together give its position  $x(y, t)$  at time  $t > 0$  without invoking the dynamics at intermediate times.

There are two ways to construct an explicit formula for  $x(y, t)$  from (5) and (6), which are discussed in the following two subsections. Both of them involve minimizing suitable functionals and can therefore be considered as (generalized) variational principles.

## 2.2 The variational principle [ERS]

Introduce in (5) and (6) a ‘mass coordinate’

$$m(y) = \int^y \rho_0(\eta) d\eta \quad (7)$$

and let

$$\begin{aligned} \Phi_0(m) &= \int^{y(m)} \eta \rho_0(\eta) d\eta, \quad U_0(m) = \int^{y(m)} u_0(\eta) \rho_0(\eta) d\eta, \\ \Phi_t(m) &= \Phi_0(m) + t U_0(m). \end{aligned} \quad (8)$$

The lower limits of integration in (7), (8) may be chosen arbitrarily. It can easily be checked that the inverse function  $y(m)$  and  $u_0(y)$  have the form

$$y(m) = \frac{d\Phi_0(m)}{dm}, \quad u_0(y(m)) = \frac{dU_0(m)}{dm}. \quad (9)$$

Hence, as long as some particle is not absorbed into a cluster, its coordinate is

$$y + u_0(y)t = \frac{d\Phi_0}{dm} + \frac{d(tU_0)}{dm} = \frac{d\Phi_t}{dm}. \quad (10)$$

However after crossing of trajectories this representation of  $x(y, t)$  breaks down. To give an expression for  $x(y, t)$  that is valid for all times, it is convenient to consider the cases of a free particle and a cluster separately.

In the new variables, condition (5) determining if a particle stays free at time  $t > 0$  takes the following form: if  $m_0 = m(y_0)$ , then for any  $m' < m_0 < m''$

$$\frac{\Phi_t(m_0) - \Phi_t(m')}{m_0 - m'} < \frac{\Phi_t(m'') - \Phi_t(m_0)}{m'' - m_0}. \quad (11)$$

In other words, at values of  $m$  corresponding to free particles the function  $\Phi_t(m)$  coincides with its convex hull  $\text{conv } \Phi_t$ , i.e., the maximal convex function not exceeding  $\Phi_t$  (Fig. 2.2). One can visualize the graph of a convex hull as the shape of an elastic thread tightly wrapped around the graph of  $\Phi_t$  from below. Since the function  $\Phi_t$  is differentiable, its convex hull is also differentiable, and at common points of their graphs the derivatives  $d\Phi_t/dm$  and  $d(\text{conv } \Phi_t)/dm$  coincide. Therefore at these points equation (10) takes the form

$$x(y, t) = \frac{d\Phi_t}{dm} = \frac{d(\text{conv } \Phi_t)}{dm}.$$

Conversely, if by the time  $t$  a group of particles initially located in the segment  $y^- < y < y^+$  forms a cluster surrounded with free particles, then condition (11) is violated at all internal points of the corresponding segment  $m^- = m(y^-) < m < m^+ = m(y^+)$  and the graph of  $\Phi_t(m)$  lies above the chord connecting  $\Phi_t(m^-)$  and  $\Phi_t(m^+)$ , and thus above the convex hull (Fig. 2.2). Moreover, (6) implies that

$$x = \frac{\Phi_t(m^+) - \Phi_t(m^-)}{m^+ - m^-} = \frac{d(\text{conv } \Phi_t)}{dm}.$$

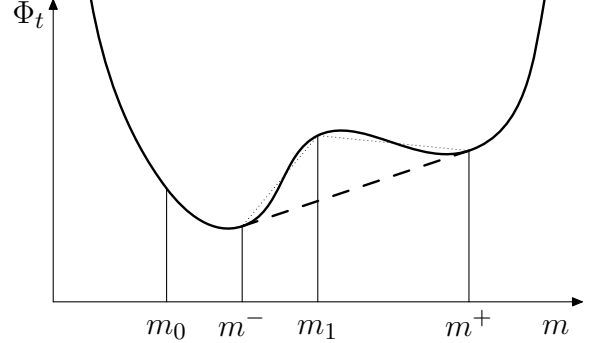


Figure 2: Construction of a solution according to the variational principle [ERS]. At  $m_0$  condition (11) is satisfied with any  $m' < m_0 < m''$ , while at  $m_1$  it is violated (e.g. for  $m' = m^-$ ,  $m'' = m^+$  the slope of the left dotted line is positive while that of the right one is negative). On the segment  $(m^-, m^+)$ , the convex hull  $\text{conv } \Phi_t(m)$  (dashed line) does not coincide with  $\Phi_t(m)$ . Thus particles corresponding to points of the segment  $(m^-, m^+)$  stick together and form a cluster of the mass  $m^+ - m^-$  by the time  $t$ .

Thus in clusters as well as in intervals of continuous mass distribution the map  $x(y, t)$  is determined by the derivative of the convex hull of  $\Phi_t$ :

$$x(y(m), t) = \frac{d(\text{conv } \Phi_t(m))}{dm}. \quad (12)$$

In [ERS], this expression is called the Generalized Variational Principle by analogy with the Hopf–Lax–Oleĭnik variational principle used to construct solutions to the Burgers equation. The latter principle itself is a variant of the least action principle in mechanics. In the present paper, (12) will be referred to as *the variational principle [ERS]*. Analogous representations for  $x(y, t)$  were introduced independently in [10] and [3].

In the case of a constant initial density, when  $\rho_0(y) \equiv 1$  and  $m \equiv y$ , the variational principle [ERS] coincides with the Hopf–Lax–Oleĭnik variational principle.

### 2.3 The variational principle [S]

Another representation for  $x(y, t)$  was introduced a decade earlier by A. I. Shnirel'man in [S]. To simplify notation, introduce the *displacement field*

$$\xi_t(y) = x(y, t) - y.$$

As long as trajectories of particles do not cross,  $\xi_t(y) = u_0(y)t$  holds and the coordinate  $x(y, t) = y + \xi_t(y)$  is monotonic as a function of  $y$ :

$$y' + \xi_t(y') \leq y'' + \xi_t(y'') \quad (13)$$

whenever  $y' < y''$ .

We consider below only displacement fields for which the integral  $\int \xi_t^2(y) \rho_0(y) dy$  exists. This is the case if the initial density  $\rho_0(y)$  and velocity  $u_0(y)$  (and therefore the displacement field) either decrease at infinity fast enough or are periodical.

Call a displacement field *feasible* if trajectories of particles do not cross, i.e., if condition (13) is satisfied for any  $y' < y''$ . According to the *variational principle* [S], the solution  $x(y, t)$  is given by

$$x(y, t) = y + \bar{\xi}_t(y),$$

where  $\bar{\xi}_t(y)$  is a feasible displacement field that minimizes the following norm of discrepancy with respect to  $u_0(y)t$ :

$$\int |\bar{\xi}_t(y) - u_0(y)t|^2 \rho_0(y) dy = \min. \quad (14)$$

In the simplest case when trajectories of particles do not cross,  $\bar{\xi}_t(y) = u_0(y)t$ ; this displacement field is feasible and therefore the discrepancy vanishes. After crossing of trajectories a feasible displacement field minimizing (14) can no longer coincide with  $u_0t$ .

Mathematically, minimization problem (14) is equivalent to the orthogonal projection of the field  $u_0(y)t$  to the set of feasible displacements, which we denoted hereafter by  $\mathcal{X}$ . Here orthogonality is understood in the sense of the functional scalar product

$$\xi \cdot \eta = \int \xi(y)\eta(y) \rho_0(y) dy,$$

and the square of the norm of discrepancy (14) is its scalar product with itself:

$$\begin{aligned} \|\xi_t - u_0t\|^2 &= \int |\xi_t(y) - u_0t|^2 \rho_0(y) dy \\ &= (\xi_t - u_0t) \cdot (\xi_t - u_0t). \end{aligned} \quad (15)$$

Observe that feasible displacements form a convex set: if displacement fields  $\xi_1(y)$ ,  $\xi_2(y)$  are feasible, then so is their 'mixture'  $(1 - \alpha)\xi_1(y) + \alpha\xi_2(y)$  for any  $0 < \alpha < 1$ . Moreover, a set of feasible displacements is closed: a limit of any sequence of feasible displacements is also feasible. It is a well-known fact of functional analysis (see, e.g., [2]) that an orthogonal projection to a closed set  $\mathcal{X}$  exists and is defined uniquely. Therefore the variational principle [S] always gives a uniquely defined result.

We now show that this result coincides with the result given by the variational principle [ERS]. Denote the displacement field defined by the map  $x(y, t)$  from (12) by

$$\tilde{\xi}_t(y(m)) = \frac{d(\text{conv } \Phi_t(m))}{dm} - y(m).$$

It suffices to check if this displacement field minimizes (14), i.e., if for any other feasible displacement  $\xi$  in  $\mathcal{X}$

$$\|\xi - u_0t\|^2 > \|\tilde{\xi}_t - u_0t\|^2.$$

It is clear from Fig. 3 that the orthogonal projection to the closed set  $\mathcal{X}$  must satisfy an even stronger inequality,

$$\|\xi - u_0t\|^2 > \|\tilde{\xi}_t - u_0t\|^2 + \|\xi - \tilde{\xi}_t\|^2, \quad (16)$$

and it is this condition that we will check.

Using (15) rewrite inequality (16) in the form

$$(\tilde{\xi}_t - \xi) \cdot (u_0t - \tilde{\xi}_t) > 0 \quad (17)$$

and compute its left-hand side:

$$\begin{aligned} &(\tilde{\xi}_t - \xi) \cdot (u_0t - \tilde{\xi}_t) \\ &= \int (\tilde{\xi}_t(y) - \xi(y))(u_0(y)t - \tilde{\xi}_t(y)) \rho_0(y) dy. \end{aligned} \quad (18)$$

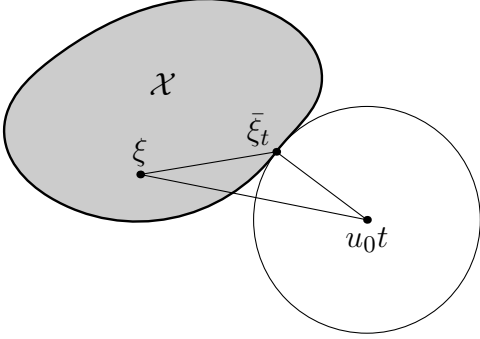


Figure 3: Orthogonal projection to a convex set  $\mathcal{X}$  in a Hilbert space. Shown is a section of the set  $\mathcal{X}$  with a hyperplane passing through the elements  $u_0 t$ ,  $\bar{\xi}_t$  and  $\xi$ . Since the point  $\bar{\xi}_t$  is closest to  $u_0 t$  in the set  $\mathcal{X}$ , this set lies outside the sphere determined by the radius  $(u_0 t, \bar{\xi}_t)$ . Therefore in the triangle connecting  $u_0 t$ ,  $\bar{\xi}_t$ , and  $\xi$  the angle at  $\bar{\xi}_t$  must be obtuse, i.e., the square of the corresponding segment is larger than the sum of squares of two other segments (inequality (16)).

Observe that

$$u_0 t - \bar{\xi}_t = y + u_0 t - x(y, t) = \frac{\partial \Phi_t}{\partial m} - \frac{\partial(\text{conv } \Phi_t)}{\partial m}, \quad (19)$$

where  $y = y(m)$ . This expression is nonzero on a set of segments  $(m_i^-, m_i^+)$  of the  $m$  axis where  $\Phi_t(m) > \text{conv } \Phi_t(m)$  is satisfied. Therefore (18) decomposes into a sum of integrals over segments  $(y(m_i^-), y(m_i^+)) = (y_i^-, y_i^+)$ , on each of which  $x(y, t)$  is constant,

$$x(y(m), t) = \frac{\Phi_t(m_i^+) - \Phi_t(m_i^-)}{m_i^+ - m_i^-} = x_i,$$

and the integrand in (18) is equal to the product of the continuous function  $(u_0(y)t - \bar{\xi}_t(y))\rho_0(y)$  with the function

$$\bar{\xi}_t(y) - \xi(y) = x_i - (y + \xi(y)),$$

which decreases because  $\xi(y)$  is a feasible displacement field. According to Bonnet's formula for the average value of the product of a monotonic and a

continuous function<sup>2</sup>, there exists a point  $y_i$  in the segment  $(y_i^-, y_i^+)$  such that

$$\begin{aligned} & \int_{y_i^-}^{y_i^+} (x_i - y - \xi(y))(u_0(y)t - \bar{\xi}_t(y))\rho_0(y) dy \\ &= (x_i - y_i^- - \xi(y_i^-)) \int_{y_i^-}^{y_i} (u_0(y)t - \bar{\xi}_t(y))\rho_0(y) dy \\ &+ (x_i - y_i^+ - \xi(y_i^+)) \int_{y_i}^{y_i^+} (u_0(y)t - \bar{\xi}_t(y))\rho_0(y) dy. \end{aligned} \quad (20)$$

Now take into account that at  $y_i^- \leq z \leq y_i^+$

$$\begin{aligned} & \int_{y_i^-}^z (u_0(y)t - \bar{\xi}_t(y))\rho_0(y) dy \\ &= \int_{m_i^-}^{m(z)} \left( \frac{\partial \Phi_t}{\partial m} - \frac{\partial(\text{conv } \Phi_t)}{\partial m} \right) dm \\ &= \Phi_t(m(z)) - \text{conv } \Phi_t(m(z)) \geq 0, \end{aligned} \quad (21)$$

and this integral vanishes only when  $z = y_i^+$ . Therefore

$$\begin{aligned} & \int_{y_i}^{y_i^+} (u_0(y)t - \bar{\xi}_t(y))\rho_0(y) dy \\ &= - \int_{y_i^-}^{y_i} (u_0(y)t - \bar{\xi}_t(y))\rho_0(y) dy, \end{aligned} \quad (22)$$

and finally

$$\begin{aligned} & \int_{y_i^-}^{y_i^+} (x_i - y - \xi(y))(u_0(y)t - \bar{\xi}_t(y))\rho_0(y) dy \\ &= (y_i^+ + \xi(y_i^+) - y_i^- - \xi(y_i^-)) \times \\ &\times (\Phi_t(m(y_i)) - \text{conv } \Phi_t(m(y_i))) > 0. \end{aligned} \quad (23)$$

Thus the whole integral (18) is positive as a sum of positive terms. Consequently, the displacement field  $\bar{\xi}_t$  satisfies the inequality (17) and minimizes the variational principle [S] (14). Conversely, the unique optimal displacement field in (14) must minimize the variational principle [ERS].

<sup>2</sup>If the function  $f(x)$  is monotonic and  $g(x)$  is continuous, then there exists a point  $c$  in the segment  $(a, b)$  such that  $\int_a^b f(x)g(x) dx = f(a) \int_a^c g(x) dx + f(b) \int_c^b g(x) dx$ .

### 3 Ballistic aggregation in cylindrically and spherically symmetric cases

Let the initial density  $\rho_0$  and velocity  $\mathbf{u}_0$  fields depend on the  $d$ -dimensional vector  $\mathbf{y}$  and be symmetric with respect to the origin:

$$\rho_0(\mathbf{y}) = \rho_0(|\mathbf{y}|), \quad \mathbf{u}_0(\mathbf{y}) = \frac{u_0(|\mathbf{y}|)}{|\mathbf{y}|} \mathbf{y}.$$

For  $d = 2$  this symmetry is cylindrical and for  $d = 3$ , spherical. The results of this section hold for any  $d > 1$ , including the nonphysical case of higher dimensions.

To extend the construction of the variational principle [ERS] to this case one has to replace the mass element  $\rho_0(y) dy$  by  $\rho_0(|\mathbf{y}|) \Omega_d d\mathbf{y}$  in (5) and (6). Here  $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$  is the (hyper)surface area of the unit sphere in  $d$ -dimensional space. Arguing similarly to Section 2.1, introducing the mass coordinate

$$m(|\mathbf{y}|) = \int_0^{|\mathbf{y}|} \rho_0(\eta) \Omega_d d\eta,$$

and setting

$$\begin{aligned} \Phi_0(m) &= \int_0^{y(m)} \eta \rho_0(\eta) \Omega_d d\eta, \\ U_0(m) &= \int_0^{y(m)} u_0(\eta) \rho_0(\eta) \Omega_d d\eta, \\ \Phi_t(m) &= \Phi_0(m) + tU_0(m), \end{aligned}$$

where

$$y(m) = \frac{d\Phi_0(m)}{dm}, \quad u_0(y(m)) = \frac{dU_0(m)}{dm},$$

we get an explicit solution that formally coincides with (12), but with a new definition of the function  $\Phi_t$ :

$$x(y(m), t) = \frac{d(\text{conv } \Phi_t(m))}{dm}.$$

### 4 On constructing solutions in the non-symmetric case

Can the above explicit construction of solutions to the ballistic aggregation problem be extended to the case of non-symmetric flows?

An existence theorem for a solution to the ballistic aggregation problem that respects the mass (1) and momentum (2) conservation laws was proved in [13]. However, this proof is non-constructive and thus gives little information about the structure of the solution and provides no method for approximating it numerically.

On the contrary, both variational principles [S] and [ERS] provide structural information and approximation procedures. Moreover, for a potential initial velocity field  $\mathbf{u}_0(\mathbf{y})$  these variational principles have natural multidimensional extensions requiring no specific assumptions on symmetry of the flow. However it is possible to construct examples of asymmetric flows for which these extensions give physically incorrect answers, as we show below.

#### 4.1 A multidimensional extension of the variational principle [ERS]

To extend the variational principle [ERS] (12) to the multidimensional case, introduce a vector “mass” coordinate  $\mathbf{m}$  of the same dimension as the vector  $\mathbf{y}$ ,

$$\left| \frac{\partial \mathbf{m}}{\partial \mathbf{y}} \right| = \rho_0(\mathbf{y}),$$

and define functions  $\Phi_0(\mathbf{m})$ ,  $U_0(\mathbf{m})$  such that analogues of equations (9) hold:

$$\mathbf{y}(\mathbf{m}) = \nabla_{\mathbf{m}} \Phi_0, \quad \mathbf{u}_0(\mathbf{m}) = \nabla_{\mathbf{m}} U_0.$$

Then it is natural to define  $\mathbf{x}(\mathbf{y}, t)$  similarly to (12):

$$\mathbf{x}(\mathbf{y}(\mathbf{m}), t) = \nabla_{\mathbf{m}} \text{conv}(\Phi_t + U_0 t). \quad (24)$$

Whatever the way in which  $\mathbf{m}(\mathbf{y})$ ,  $\Phi_0(\mathbf{m})$  and  $U_0(\mathbf{m})$  are defined in general, for the uniform initial density  $\rho_0(\mathbf{y}) \equiv 1$  it suffices to set

$$\mathbf{m} = \mathbf{y}, \quad \Phi_0(\mathbf{m}) = \frac{|\mathbf{m}|^2}{2} \quad (25)$$

and take  $U_0$  for the initial velocity field potential  $\mathbf{u}_0$ . Still even in this case it is possible to show that the proposed modification of the variational principle [ERS] leads to incorrect results.

Consider a spherical wave expanding from the origin. Suppose that initially matter is at rest while the velocity potential  $U_0$  is constant everywhere except a small area about the origin. Asymptotically for large times the size of this area and details of the velocity field inside it become inessential and the initial velocity potential can be approximated with

$$U_0(\mathbf{y}) = \begin{cases} 0, & \mathbf{y} = 0, \\ U > 0, & |\mathbf{y}| > 0. \end{cases}$$

At time  $t > 0$  the radius  $R(t)$  of the spherical wave is defined by the derivative  $d(\text{conv } \Phi_t)/d|\mathbf{y}|$  at  $|\mathbf{y}| = 0$ , where

$$\begin{aligned} \Phi_t(\mathbf{y}) &= \frac{|\mathbf{y}|^2}{2} + U_0(\mathbf{y})t, \\ \text{conv } \Phi_t(\mathbf{y}) &= \begin{cases} \sqrt{2Ut}|\mathbf{y}|, & 0 \leq |\mathbf{y}| \leq \sqrt{2Ut}, \\ \frac{|\mathbf{y}|^2}{2} + Ut, & \sqrt{2Ut} \leq |\mathbf{y}|. \end{cases} \end{aligned} \quad (26)$$

So the proposed extension of the variational principle [ERS] predicts that the radius of the spherical wave grows as  $\sqrt{2Ut}$  regardless of the dimension of space.

On the other hand, the full mass of the wave front at time  $t$  is equal to the mass of matter initially distributed over the area contained within the wave front, i.e., proportional to  $R^d$ , where  $d = 2$  in the cylindrical case and  $d = 3$  in spherical case. Since matter moves only in radial directions, the full momentum is preserved in any cone with the vertex at the origin. Hence the product of the velocity of the wave front by its mass must stay constant:  $R^d dR/dt = \text{const}$ , so that

$$R(t) \propto t^{1/(d+1)}. \quad (27)$$

It is this time dependence of  $R$  that appears in Section 3, while (26) and (24) turn out to hold only for  $d = 1$ .

It is interesting to determine the time dependence of the kinetic energy  $E(t)$ . The full kinetic energy of the flow is the product of the mass of the wave front

by its squared radial velocity. Accordingly, in the physically correct solution (27) we have  $E \propto R^{-d} \propto t^{-d/(d+1)}$ , while formula (24) gives  $E \propto t^{(d-2)/2}$ ; for  $d = 3$ , the kinetic energy of the spherical wave turns out to grow indefinitely!

Although the choice of mass coordinates  $\mathbf{m}(\mathbf{y})$  is not uniquely determined by  $\rho_0$  alone, this freedom is not enough to resolve this contradiction. For a constant initial density  $\rho_0$  one can show [4] that the only way to introduce mass coordinates while preserving the spherical symmetry is (25). Thus the convex hull construction of the variational principle [ERS] cannot be extended to the non-symmetric multidimensional case.

Another analysis showing incorrectness of this extension of the variational principle [ERS] is given in [12].

## 4.2 A multidimensional extension of the variational principle [S]

To extend the variational principle [S] to the multidimensional case, one has to define a set of feasible displacement fields in such a way that a solution to the ballistic aggregation problem would still be defined by an orthogonal projection construction.

In one dimension a displacement field  $\xi(y)$  is feasible when the function  $x(y) = y + \xi(y)$  is monotonic, or equivalently when its primitive function is convex. Observe that, regardless of the dimension, the ‘mixture’ of two convex functions  $\Phi, \Psi$  of the form  $(1 - \alpha)\Phi + \alpha\Psi$ , where  $0 \leq \alpha \leq 1$ , is always convex and that the differentiation operation is linear. Therefore the class of gradients of convex functions in any dimension is a convex subset of a suitable functional space. Call the displacement field  $\xi(\mathbf{y}) = \mathbf{x}(\mathbf{y}) - \mathbf{y}$  *feasible* if  $\mathbf{x}(\mathbf{y})$  is the gradient of a convex function. Then a natural multidimensional extension of the variational principle [S] consists in minimizing the norm of discrepancy

$$\int |\xi_t(\mathbf{y}) - \mathbf{u}_0(\mathbf{y})t|^2 \rho_0(\mathbf{y}) d\mathbf{y} = \min \quad (28)$$

over the class of feasible displacement fields defined as above.



Suppose that the initial velocity field  $\mathbf{u}_0$  is potential, so that

$$\mathbf{u}_0(\mathbf{y}) = \nabla U_0(\mathbf{y}), \quad \mathbf{x}(\mathbf{y}, t) = \mathbf{y} + \nabla U_0(\mathbf{y})t = \nabla \Phi_t(\mathbf{y}),$$

where

$$\Phi_t(\mathbf{y}) = \frac{1}{2}|\mathbf{y}|^2 + U_0(\mathbf{y})t.$$

Evidently, at  $t = 0$  the function  $\Phi_0(\mathbf{y}) = |\mathbf{y}|^2/2$  is convex.

As long as the Jacobian of the map  $\mathbf{y} \mapsto \mathbf{x}(\mathbf{y}, t)$  is non-zero (positive), trajectories of particles do not cross. Since this Jacobian coincides with the determinant of the matrix of second derivatives (the Hessian matrix)  $(\partial_i \partial_j \Phi_t)$ , convexity of the function  $\Phi_t(\mathbf{y})$  is preserved. Therefore the displacement field  $\mathbf{u}_0(\mathbf{y})t$  stays feasible and the discrepancy norm (28) is equal to zero, so the proposed form of the variational principle [S] gives a correct solution in this case.

After crossing of two infinitesimally close trajectories the determinant of the Hessian matrix vanishes, the function  $\Phi_t(\mathbf{y})$  ceases to be convex, and the displacement field minimizing the (28) can no longer be given by  $\mathbf{u}_0 t$ . Similarly to the argument of Section 2.3 one can show that in the spherically symmetric case the proposed form of the variational principle [S] (28) gives a correct solution. However the following example shows that the proposed extension gives wrong results in less symmetrical cases.

Consider two identical cylindrical waves in a two-dimensional space with centers separated from each other by  $2r$  and radii growing as  $R(t) = Kt^{1/3}$  in accordance with (27). At time  $t > (r/K)^3$  the fronts of these waves start to coalesce, giving rise to a flat portion of the wave front with matter moving longitudinally (Fig. 4). We now construct the displacement field describing the distribution of matter in the cylindrical portions and in the flat portion of the wave front for  $t > (r/K)^3$ .

Let the coordinates be chosen so that centers of waves are at  $(\pm r, 0)$ . At time  $t$  the flat portion is the segment of length  $2l(t) = 2\sqrt{R^2(t) - r^2}$  in the  $y$  axis. Matter coming to this segment from the rounded portions of the wave front sticks together and moves outward with velocity  $v = \dot{R} \cdot l/R$ . Observe that the

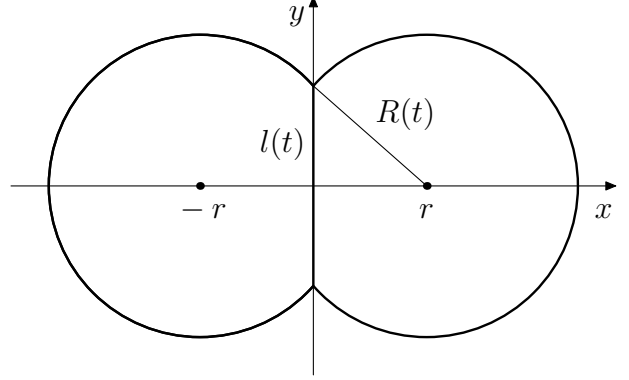


Figure 4: Two cylindrical waves: an example of a flow for which the proposed multidimensional extension of the variational principle [S] (28) gives an incorrect result. The front is shown at time  $t$  satisfying  $(r/K)^3 < t < t^* = (5/2)^{5/2}(r/K)^3$ .

length of the segment increases faster, at the rate

$$\dot{l} = \frac{R\dot{R}}{\sqrt{R^2 - r^2}} = \dot{R} \cdot \frac{R}{l} > \dot{R} \cdot \frac{l}{R} = v;$$

thus endpoints of the flat segment do not affect the motion of matter inside it.

Let  $T(l)$  be the inverse function for  $l(t)$ ,

$$T(l) = \frac{1}{K^3}(l^2 + r^2)^{3/2},$$

and take  $l$  as a parameter. Since  $R(t) = Kt^{1/3}$ , we have  $\dot{R}/R = 1/3t$ , so that  $v(l) = l\dot{R}/R = l/3T(l)$ . Therefore at time  $t > T(l)$  the particle parametrized with  $l$  will be located at

$$y(l, t) = l + v(l)(t - T(l)) = \frac{2}{3}l + \frac{tl}{3T(l)}.$$

The distribution of mass inside the flat segment stays continuous as long as the map  $l \mapsto y(l, t)$  is monotonic, i.e., while  $\partial y / \partial l = 2/3 + (t/3)(l/T(l))' > 0$  for all  $l$ . This inequality is satisfied for  $(r/K)^3 < t < t^*$ , where

$$t^* = \min_{l > r/\sqrt{2}} \left( -\frac{2}{(l/T(l))'} \right) = \left( \frac{5}{2} \right)^{5/2} \left( \frac{r}{K} \right)^3.$$

Here minimization is performed for  $l > r/\sqrt{2}$  because for  $0 \leq l \leq r/\sqrt{2}$  the condition  $\partial y/\partial l \geq 0$  holds for any  $t$ .

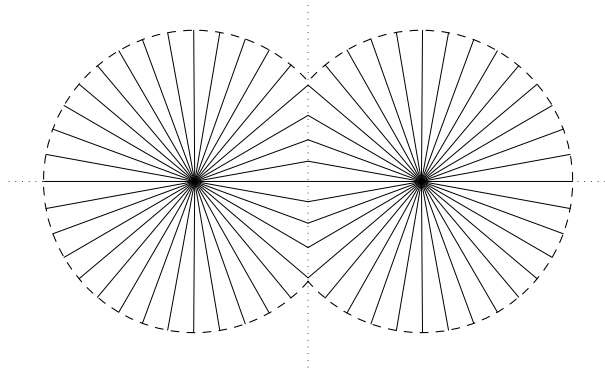


Figure 5: Continuous black lines starting at  $(\pm r, 0)$  show initial positions of particles located at the points of the wave front in the flow of Fig. 4 (for times  $(r/K)^3 < t < t^* = (5/2)^{5/2}(r/K)^3$ ). Note that particles forming the flat portion of the front are initially located on two-segment broken lines in the central area of the figure.

Observe now that initial positions of particles that are located at the point  $(0, y(l, t))$  in the flat segment of the front at time  $T(l) \leq t < t^*$  form a two-segment broken line connecting the point  $(0, l)$  with centers of both waves (Fig. 5). On the other hand, the preimage of any point in a mapping defined by the gradient of a convex function must itself be a convex set (a point, a segment, or a convex domain). Since preimages of some points in the displacement field are non-convex broken lines, this displacement field cannot be realized by the gradient of a convex function, i.e., it is not ‘feasible’ in the above sense and the proposed natural extension of the variational principle [S] provides an incorrect description of describes the motion of matter in the case of two cylindrical waves.

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